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Non-Metric Gravity II: Spherically Symmetric Solution, Missing Mass and Redshifts of Quasars

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We continue the study of the non-metric theory of gravity introduced in hep-th/0611182 and gr-qc/0703002 and obtain its general spherically symmetric vacuum solution. It respects the analog of the Birkhoff theorem, i.e., the vacuum spherically symmetric solution is necessarily static. As in general relativity, the spherically symmetric solution is seen to describe a black hole. The exterior geometry is essentially the same as in the Schwarzschild case, with power-law corrections to the Newtonian potential. The behavior inside the black-hole region is different from the Schwarzschild case in that the usual spacetime singularity gets replaced by a singular surface of a new type, where all basic fields of the theory remain finite but metric ceases to exist. The theory does not admit arbitrarily small black holes: for small objects, the curvature on the would-be horizon is so strong that non-metric modifications prevent the horizon from being formed.

The theory allows for modifications of gravity of very interesting nature. We discuss three physical effects, namely, (i) correction to Newton's law in the neighborhood of the source, (ii) renormalization of effective gravitational and cosmological constants at large distances from the source, and (iii) additional redshift factor between spatial regions of different curvature. The first two effects can be responsible, respectively, for the observed anomaly in the acceleration of the Pioneer spacecraft and for the alleged missing mass in spiral galaxies and other astrophysical objects. The third effect can be used to propose a non-cosmological explanation of high redshifts of quasars and gamma-ray bursts.

I. INTRODUCTION

This is the second paper in the series devoted to the detailed study of the non-metric theory of gravity introduced in [1, 2]. As in the case of general relativity (GR), one of the first applications of any theory of gravity must be the description of the geometry of a strongly gravitating object — a black hole. This is the basic aim of the present paper, in which we obtain and analyze the spherically symmetric vacuum solution. The considerations of this work will also serve as an illustration to the rather abstract formalism of [2].

The theory of gravity under investigation is defined by the action

$$S[B, A, \Psi] = \frac{1}{8\pi G} \int_M B^i \wedge F^i + \frac{1}{2} \left(\Psi^{ij} + \phi\, \delta^{ij} \right) B^i \wedge B^j\,, \qquad \phi = \phi\left[\mathrm{Tr}\left(\Psi^2 \right), \mathrm{Tr}\left(\Psi^3 \right) \right]\,. \tag{1}$$

Here, B^i is a (complex) $\mathfrak{su}(2)$ Lie-algebra valued two-form (the indices $i, j = 1, 2, 3$ belong to the $\mathfrak{su}(2)$ Lie algebra), $F^i = dA^i + \frac{1}{2}[A, A]^i$ is the curvature of the $\mathfrak{su}(2)$ Lie-algebra valued connection A^i, Ψ^{ij} is a traceless symmetric "Lagrange multiplier" field, and G is the Newton's constant. The function $\phi\left[\mathrm{Tr}\left(\Psi^2\right), \mathrm{Tr}\left(\Psi^3\right)\right]$ is a function of two scalar invariants that can be constructed from Ψ^{ij} and is responsible for the departure of the theory under consideration from GR. The constant part of the function ϕ is (a multiple of) the usual cosmological constant Λ.

The theory with action (1) modifies the Plebański self-dual formulation of general relativity [3], in which $\phi \equiv \phi_0 = -\Lambda/3$. In the Plebański theory, the field B^i, according to the equations of motion, can be canonically decomposed into products of tetrad basis one-forms, which then define a unique distinguished metric satisfying the Einstein equation. With the appearance of a nontrivial function $\phi(\Psi)$ in (1), this property is no longer valid: the arising metric is defined only up to a conformal factor, and it does not, in general, satisfy the vacuum Einstein equations.

The nature of this modification of gravity is quite special and deserves a few words. In the numerous existing schemes of modified gravity, one usually deals with a metric theory and modifies the Hilbert–Einstein action by introducing extra degrees of freedom — either by increasing the number of derivatives (higher-derivative gravity), or by introducing extra fields (scalar-vector-tensor theories), or by considering extra dimensions (braneworlds). Nothing of the listed takes place in our generalization: the theory remains four-dimensional, there

are no extra fields, the number of derivatives is not increased, and, as in GR, there are still just two propagating degrees of freedom, as can be seen from the canonical analysis of the theory; see, e.g., [4]. This is achieved by first recasting the theory in a form which does not contain any metric [3], and then modifying the theory in this non-metric form. It is thus important to stress that the term "non-metric" is used here not in the sense that some additional degrees of freedom are present along with the usual metric, but in the sense that metric does not even appear in the formulation of the theory.

The difference between our modified gravity and a variety of other approaches can also be seen from the fact that our theory, while modifying the spherically symmetric solution, does not modify the formal general-relativistic cosmological equations. Indeed, due to the high symmetry, the field Ψ, which is a close analog of the Weyl curvature spinor in our theory, is identically zero for cosmological solutions, which then coincide with those of general relativity with a cosmological constant determined by $\phi(0)$. However, any departures from homogeneity and isotropy will be essential, so that the evolution of perturbations in our theory will, in principle, be different from that in the concordance LCDM cosmology. This issue will be studied separately.

A specific choice of the function ϕ uniquely fixes a theory from the class (1). One way of fixing the form of this function is to regard it as an effective quantum contribution to the original classical action. In this respect, it was argued in [1] that the class of theories (1) is closed under the renormalization-group flow. The conjecture of asymptotic safety applied to the case at hand then asserts the existence of a non-trivial ultra-violet fixed point of the renormalization-group flow described by a certain function $\phi^*(\Psi)$. It would then make sense to choose this fixed-point function $\phi^*(\Psi)$ in (1) because the *quantum* field theory so defined would have extremely appealing properties: the action would not get perturbatively renormalized, describing an essentially finite quantum theory of gravity. This gives a possible scenario for fixing $\phi(\Psi)$ from the theoretical side.

In a pure quantum theory of gravity, the Planck scale is the only scale that can enter the renormalized action. However, once the theory is coupled to matter, the corresponding quantum loops in Feynman diagrams will also affect the function ϕ. Thus, it should be expected that the ultra-violet fixed-point function ϕ^* will contain more than one physical scale. Qualitative features of the function ϕ^* can be anticipated from the fact that, as we shall see in this paper, a theory with a non-constant function $\phi(\Psi)$ in many respects be-

haves like a theory with physical parameters depending on the scale. Indeed, as we shall see, the function $\phi(\Psi)$ itself will receive an interpretation of a curvature-dependent cosmological "constant". Another effect is that passing between spatial regions of different curvature generally introduces a renormalization in the strength of the gravitational interaction. All this is strongly suggestive of the renormalization-group flow phenomena, where one has to identify Ψ (having the dimension of the curvature) with the energy scale squared. The crucial difference between our scheme and the renormalization-group flow familiar from the framework of effective field theory is that similar effects arise here in a diffeomorphism-invariant context. This renormalization-group interpretation suggests that the function $\phi^*(\Psi)$ must look as a sequence of plateaus, with crossover regions between plateaus corresponding to the scales where new physics (new degrees of freedom) come into play. This discussion motivates some assumptions we make about the form of the function $\phi(\Psi)$ in the section where we discuss possible long distance modifications of gravity.

Of special importance is the question of coupling our theory of gravity (1) to other fields. As was discussed in [2], coupling to Yang-Mills fields (or electromagnetic field) is seamless, and the action described in [5] extends to the non-metric situation without any problem. This action tells us how massless particles such as photons interact with gravity and predicts their motion in a non-metric background. It is found that, in the approximation of geometric optics, photons move along null geodesics of the metric determined by the field B^i. The ambiguity in the choice of the conformal factor discussed in detail in [2] does not affect this conclusion because the paths of null geodesics (but, of course, not the affine parameter along them) are independent of this choice. An important issue that has not yet been addressed in the framework of non-metric gravity and that prevents us from a complete analysis of the physical predictions of theory (1) concerns coupling to (massive) matter degrees of freedom. The action for a massive field of spin 1/2 proposed in [5] for the Plebański formulation of general relativity turns out to be incompatible with the non-metric character of the theory under investigation, hence, calls for revision.

Quantitative predictions of our theory, e.g., concerning the motion of stars and gas in galaxies, will depend on the details of the coupling of massive matter to the basic gravitational degrees of freedom. However, its certain general features can already be described in the absence of these details. For instance, in the domains of "metricity," in which the function ϕ is almost constant (where its dimensionless derivatives $|\partial\phi/\partial\Psi| \ll 1$), the present

theory of gravity behaves very closely to general relativity, and it is quite reasonable to expect that matter will also behave accordingly, moving relativistically in the background of the arising metric. Assuming that several such domains of "metricity" exist at different scales of curvatures, we find that the effective gravitational mass of a central body is different in the corresponding spatial regions. From this simple observation one can conclude that the effective gravitational mass of a body continuously depends on the distance to this body even in the case of general function $\phi(\Psi)$ — the effect of scale-dependence of the gravitational coupling which was mentioned above. One can use this property to account for the phenomenon of missing mass observed in gravitating objects such as spiral and elliptical galaxies. Another interesting general prediction of the present theory is the appearance of an additional redshift factor between regions of different space-time curvature. As we point out in this paper, this effect can be used to account for the observed high redshifts of quasars and gamma-ray bursts. However, practical use of these features of the new theory to explain physical phenomena requires the specific knowledge of the underlying function $\phi(\Psi)$ together with the analysis of the physical content and interpretation of the theory. This will be the subject of the future work.

The organization of the paper is as follows. To obtain a spherically symmetric solution, we first obtain an ansatz for the symmetric field B^i. This is done in Sec. II, in which we also analyze consequences of the modified "metricity" equations. The field equations are obtained in Sec. III. Solutions are analyzed and interpreted in Sec. IV. Possible modifications of gravity are discussed in Sec. V, and possible physical effects in Sec. VI. Our results are summarized in Sec. VII. In the appendix, we give a detailed proof of the static property of the spherically symmetric solution in the theory under consideration and discuss in more generality some details of the large-distance modifications of gravity.

II. SPHERICALLY SYMMETRIC ANSATZ AND METRICITY EQUATIONS

A. Spherically symmetric Lie-algebra-valued two-form

The most general spherically symmetric $\mathfrak{su}(2)$ Lie-algebra-valued two-form can be obtained from the condition that an SO(3) rotation corresponds to a gauge transformation. There exists standard technique in the literature allowing one to obtain the relevant expres-

sion; see, e.g., [6]. One gets:

$$
\begin{aligned}
B \equiv \sum_i B^i \tau^i \;&=\; (\phi_1 dt \wedge d\theta - \chi_1 \sin\theta \, dr \wedge d\phi)\, \tau^1 \\
&+\; (\phi_2 \sin\theta \, dt \wedge d\phi + \chi_2 dr \wedge d\theta)\, \tau^2 \\
&+\; (\phi_3 \sin\theta \, d\theta \wedge d\phi + \chi_3 dt \wedge dr)\, \tau^3 \,.
\end{aligned}
\tag{2}
$$

Here, (t, r, θ, ϕ) is the standard set of spherical coordinates, and ϕ_i, χ_i, $i = 1, 2, 3$, are functions of t and r only. The symbols τ^i denote the $\mathfrak{su}(2)$ generators $\tau^i = -(i/2)\sigma^i$, where σ^i are the Pauli matrices. As it turns out, it is much more convenient to work not with the adjoint, but with the fundamental representation of $\mathfrak{su}(2)$. This amounts to working in the spinor formalism. We used spinor formalism rather heavily in [2], and will continue to do so in this paper. To pass from the SO(3) form of the fields to their spinor representation one has to replace every lower-case Latin index $i, j, \ldots = 1, 2, 3$ by a symmetric pair of unprimed spinor indices. Equivalently, every $\mathfrak{su}(2)$ Lie-algebra-valued field gets replaced by a 2×2 matrix-valued field. Thus, it will be convenient to rewrite the above ansatz for B in terms of the matrices

$$
\tilde{X}_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{X}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{X} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}
$$

which are related to τ^i as $\tau^1 = (1/2i)(\tilde{X}_+ - \tilde{X}_-)$, $\tau^2 = (1/2)(\tilde{X}_+ + \tilde{X}_-)$, $\tau^3 = -i\tilde{X}$. We have:

$$
\begin{aligned}
B \;&\equiv\; \tilde{X}_- B_- + \tilde{X}_+ B_+ + \tilde{X} B_0 \\
&=\; \tilde{X}_- \left[\left(-\frac{1}{2i}\phi_1 dt + \frac{1}{2}\chi_2 dr \right) \wedge d\theta + \left(\frac{1}{2i}\chi_1 dr + \frac{1}{2}\phi_2 dt \right) \sin\theta \wedge d\phi \right] \\
&+\; \tilde{X}_+ \left[\left(\frac{1}{2i}\phi_1 dt + \frac{1}{2}\chi_2 dr \right) \wedge d\theta + \left(-\frac{1}{2i}\chi_1 dr + \frac{1}{2}\phi_2 dt \right) \sin\theta \wedge d\phi \right] \\
&-\; i\tilde{X} \left(\phi_3 \sin\theta \, d\theta \wedge d\phi + \chi_3 dt \wedge dr \right) \,.
\end{aligned}
\tag{4}
$$

It is not hard to show, and this fact was used heavily in [2], that, by a convenient choice of the spinor basis, the spinor counterpart of the quantity Ψ can be put into the form

$$
\Psi = \alpha \left(\tilde{X}_- \otimes \tilde{X}_- + \tilde{X}_+ \otimes \tilde{X}_+ \right) + \beta \left(\tilde{X}_+ \otimes \tilde{X}_- + \tilde{X}_- \otimes \tilde{X}_+ + 4\tilde{X} \otimes \tilde{X} \right) \,. \tag{5}
$$

The functions α and β are related to the invariant characteristics of Ψ:

$$
\mathrm{Tr}\left(\Psi^2\right) = 2\alpha^2 + 6\beta^2 \,, \qquad \mathrm{Tr}\left(\Psi^3\right) = 6\beta\left(\alpha^2 - \beta^2\right) \,. \tag{6}
$$

In view of (6), one can regard $\phi\left[\mathrm{Tr}\left(\Psi^2\right), \mathrm{Tr}\left(\Psi^3\right)\right]$ as a function of α and β.

In the spherically symmetric case, the field Ψ^{ij} has the form

$$\Psi^{ij} = \psi(r)\left(x^i x^j - \frac{1}{3}\delta^{ij} r^2\right) , \tag{7}$$

where x^i, $i = 1, 2, 3$, are the natural Euclidean coordinates realizing the group of rotations, and $r^2 = \sum_i \left(x^i\right)^2$. This condition implies $\alpha = 0$ in (5), and the field Ψ is parameterized by a single function β :

$$\Psi = \beta\left(\tilde{X}_+ \otimes \tilde{X}_- + \tilde{X}_- \otimes \tilde{X}_+ + 4\tilde{X} \otimes \tilde{X}\right) . \tag{8}$$

In other words, the Lagrange multiplier field Ψ must be algebraically special, of type D. This is, of course, exactly the property of Ψ in a spherically symmetric solution of the usual GR. This property remains unchanged in the non-metric theory of gravity under consideration.

B. "Metricity" equations

With the above ansatz for the B field, it is easy to compute the quantity $B^i \wedge B^j$ that appears in the metricity equations — the equations stemming from (1) as the field Ψ is varied. These equations are discussed in [2] in great length, and we will not repeat that discussion here. We just note that, as it is easy to check, the quantity $B^i \wedge B^j$, with B^i given by (2) above, is diagonal as a 3×3 matrix. This implies that the matrix $\Phi := \partial\phi/\partial\Psi$, which is the traceless part of $B^i \wedge B^j$ in view of the metricity equation, is also diagonal. Hence, $\phi_\alpha := \partial\phi/\partial\alpha$ must be identically zero in view of equation (26) of [2]. The property $\alpha = 0$ ensures this condition, in particular, if the function ϕ is a regular function of its arguments (6) in the neighborhood of zero.

Let us now write the metricity equations specialized to the type D at hand. They are easy to obtain from equations (29) of [2] by setting $\phi_\alpha = 0$. We have

$$\begin{aligned} &B_+ \wedge B_+ = B_- \wedge B_- = B_+ \wedge B_0 = B_- \wedge B_0 = 0 , \\ &2B_+ \wedge B_- + B_0 \wedge B_0 = -2\phi_\beta\left(B_+ \wedge B_- - \frac{1}{4}B_0 \wedge B_0\right) , \end{aligned} \tag{9}$$

where $\phi_\beta := \partial\phi/\partial\beta$. Let us now see what this implies about our ansatz (4). The last two equations in the first line of (9) are automatically satisfied, while the first two equations imply $\phi_1\chi_1 = \phi_2\chi_2$. This allows us to write the $\tilde{X}_\pm$ components in (4) as

$$B_- = c\, m \wedge l , \qquad B_+ = c\, n \wedge \bar{m} , \tag{10}$$

where

$$m := \xi\left(-\frac{\phi_1}{\phi_2}d\theta + i\sin\theta\, d\phi\right)\,, \qquad l := \eta\left(-\frac{1}{2i}\phi_2 dt + \frac{1}{2}\chi_1 dr\right)\,,$$
$$\bar{m} := \xi\left(-\frac{\phi_1}{\phi_2}d\theta - i\sin\theta\, d\phi\right)\,, \qquad n := \eta\left(-\frac{1}{2i}\phi_2 dt - \frac{1}{2}\chi_1 dr\right)\,. \tag{11}$$

In these expressions, ξ and η are arbitrary functions of t and r, and $c = \xi\eta$. To fix these functions, we equate the component B_0 in (4) to the third canonical two-form:

$$-\,i\left(\phi_3 \sin\theta\, d\theta\wedge d\phi + \chi_3 dt\wedge dr\right) = l\wedge n - m\wedge\bar{m}\,, \tag{12}$$

which gives ξ^2 and η^2 in terms of ϕ_i and χ_i :

$$\xi^2 = \frac{\phi_2\phi_3}{2\phi_1}\,, \qquad \eta^2 = \frac{2\chi_3}{\phi_2\chi_1}\,. \tag{13}$$

Eventually, the expression for B takes the form

$$B = \tilde{X}_- c\, m\wedge l + \tilde{X}_+ c\, n\wedge\bar{m} + \tilde{X}\left(l\wedge n - m\wedge\bar{m}\right)\,. \tag{14}$$

The last metricity equation [the second line of (9)] then relates the function c to the function β in (8) through a derivative of ϕ :

$$c^2 = \frac{1-\phi_\beta/2}{1+\phi_\beta}\,. \tag{15}$$

Now we are going to simplify the expressions for one-forms (11). First, we can choose ξ as a new radial coordinate. After this, introducing new functions f, g, and h, one can write the one-forms l, n, m, and $\bar{m}$ as

$$l = \frac{1}{\sqrt{2}}\left(f dt - g dr\right)\,, \quad n = \frac{1}{\sqrt{2}}\left(f dt + g dr\right)\,, \quad m,\,\bar{m} = \frac{r}{\sqrt{2}}\left(h d\theta \pm i\sin\theta\, d\phi\right)\,. \tag{16}$$

By solving the system of field equations, one can prove that the function h is just a constant, and that the sought functions f, g, and β are independent of time. We demonstrate this property in the appendix. Then, by rescaling the angle ϕ and the radial coordinate r, we can set h to be identically equal to unity, after which the canonical set of one-forms is expressed as

$$l = \frac{1}{\sqrt{2}}\left(f dt - g dr\right)\,, \quad n = \frac{1}{\sqrt{2}}\left(f dt + g dr\right)\,, \quad m,\,\bar{m} = \frac{r}{\sqrt{2}}\left(d\theta \pm i\sin\theta\, d\phi\right)\,, \tag{17}$$

and the metric $ds^2 = 2l\otimes n - 2m\otimes\bar{m}$ defined by tetrad (17) assumes the standard form

$$ds^2 = f^2 dt^2 - g^2 dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{18}$$

of the spherically symmetric problem of general relativity. In this form, we assume all space-time coordinates to be real. Thus, the only novelty as compared to the GR case is the presence of the function c in the first two terms of (14). The function c is related to β through the derivative $\phi_\beta(\beta)$ of $\phi(\beta)$ via (15). Note that, if $\phi_\beta = 0$, we have $c = 1$, and the two-form B reduces to that of the GR (with a cosmological constant determined by the constant value of ϕ).

III. FIELD EQUATIONS

The theory under consideration respects the analog of Birkhoff's theorem. The proof of the static property of the metric is given in the appendix; its asymptotic flatness is demonstrated below. In the main text, we just assume the tetrad forms to be given by expression (17), and the Ψ field by expression (8), with f, g, and β being functions of the radial coordinate r only, and we will be looking for solutions of these functions.

A. Structural equations

The Cartan structural equations (called compatibility equations in [2]) give an algebraic relation between the connection form A and the two-form B and its exterior derivative, which allows one to solve these equations with respect to A. As the first step towards this solution, we obtain manageable expressions for the quantities $dB_\pm$ and dB_0, where

$$B_- = c\, m \wedge l\,, \qquad B_+ = c\, n \wedge \bar{m}, \qquad B_0 = l \wedge n - m \wedge \bar{m}\,, \tag{19}$$

and the one-forms l, n, m, $\bar{m}$ are given by (17). This is an easy exercise in differentiation, similar to what one does in obtaining the Ricci rotation coefficients for metric (18). After simple calculation, we obtain

$$\begin{aligned} dB_- &= \frac{1}{\sqrt{2}r}\left[\frac{(rcf)'}{gf}\, l \wedge n \wedge m - c \cot\theta\, l \wedge m \wedge \bar{m}\right], \\ dB_+ &= -\frac{1}{\sqrt{2}r}\left[\frac{(rcf)'}{gf}\, l \wedge n \wedge \bar{m} + c \cot\theta\, n \wedge m \wedge \bar{m}\right], \\ dB_0 &= \frac{\sqrt{2}}{rg}\,(l - n) \wedge m \wedge \bar{m}\,, \end{aligned} \tag{20}$$

where the prime denotes differentiation with respect to r.

Another way of expressing the results for $dB_\pm$, dB_0 is to project the arising three-forms onto the basis of dual one-forms. This is conveniently done by introducing, for an arbitrary four-form C, the scalar quantity C_v defined as

$$C = C_v \, l \wedge n \wedge m \wedge \bar{m} \, . \tag{21}$$

We get

$$\begin{aligned} (dB_- \wedge \bar{m})_v &= (dB_+ \wedge m)_v = \frac{(rcf)'}{\sqrt{2}\, rgf} \, , \\ -\,(dB_- \wedge n)_v &= (dB_+ \wedge l)_v = \frac{c}{\sqrt{2}\, r} \cot\theta \, , \\ (dB_0 \wedge n)_v &= (dB_0 \wedge l)_v = \frac{\sqrt{2}}{rg} \, , \end{aligned} \tag{22}$$

with all other components being zero.

The Cartan structural equations that determine the components $A_\pm$, A_0 of the connection have the form

$$\begin{aligned} C_- &= A_- \wedge B_0 - A_0 \wedge B_- \, , \\ C_+ &= A_0 \wedge B_+ - A_+ \wedge B_0 \, , \\ C_0 &= A_- \wedge B_+ - A_+ \wedge B_- \, , \end{aligned} \tag{23}$$

with

$$C_\pm = -dB_\pm \, , \qquad C_0 = -\frac{1}{2} dB_0 \, ; \tag{24}$$

see [2] for derivation. These equations are obtained from the equations in [2] by specializing to the case of a “constant” basis in the space of “internal” spinors, see [2] for a description of the distinction between “internal” and “spacetime” spinors. The basis in which the Lagrange multiplier has the form (8) is precisely such a basis.

These equations are solved by first computing their components using definition (21), which gives

$$\begin{aligned} (C_- \wedge l)_v &= (A_-)_n \, , & (C_- \wedge n)_v &= -(A_-)_l + c(A_0)_{\bar{m}} \, , \\ (C_- \wedge m)_v &= -(A_-)_{\bar{m}} \, , & (C_- \wedge \bar{m})_v &= (A_-)_m - c(A_0)_n \, , \\[4pt] (C_+ \wedge l)_v &= c(A_0)_m - (A_+)_n \, , & (C_+ \wedge n)_v &= (A_+)_l \, , \\ (C_+ \wedge m)_v &= -c(A_0)_l + (A_+)_{\bar{m}} \, , & (C_+ \wedge \bar{m})_v &= -(A_+)_m , \\[4pt] (C_0 \wedge l)_v &= c(A_-)_m \, , & (C_0 \wedge n)_v &= c(A_+)_{\bar{m}} \, , \\ (C_0 \wedge m)_v &= -c(A_-)_l \, , & (C_0 \wedge \bar{m})_v &= -c(A_+)_n \, , \end{aligned} \tag{25}$$

and then solving this linear system of equations, with the result

$$
\begin{aligned}
&(A_-)_l = -c^{-1}(C_0 \wedge m)_v\,, && (A_-)_n = (C_- \wedge l)_v\,,\\
&(A_-)_m = c^{-1}(C_0 \wedge l)_v\,, && (A_-)_{\bar m} = -(C_- \wedge m)_v\,,\\
&(A_+)_l = (C_+ \wedge n)_v\,, && (A_+)_n = -c^{-1}(C_0 \wedge \bar m)_v\,,\\
&(A_+)_m = -(C_+ \wedge \bar m)_v\,, && (A_+)_{\bar m} = c^{-1}(C_0 \wedge n)_v\,,\\
&(A_0)_l = c^{-2}(C_0 \wedge n)_v - c^{-1}(C_+ \wedge m)_v\,, && (A_0)_n = c^{-2}(C_0 \wedge l)_v - c^{-1}(C_- \wedge \bar m)_v\,,\\
&(A_0)_m = -c^{-2}(C_0 \wedge \bar m)_v + c^{-1}(C_+ \wedge l)_v\,, && (A_0)_{\bar m} = -c^{-2}(C_0 \wedge m)_v + c^{-1}(C_- \wedge n)_v\,.
\end{aligned}
\tag{26}
$$

Here, the subscripts l, n, m, and $\bar m$ indicate the corresponding components in the development over the basis one-forms (17).

Using (22) and (24), we finally have

$$
\begin{gathered}
A_- = -\frac{1}{\sqrt{2}\,rcg}\,m\,, \qquad A_+ = -\frac{1}{\sqrt{2}\,rcg}\,\bar m\,,\\
A_0 = \frac{1}{\sqrt{2}\,g}\left[\frac{(rcf)'}{rcf} - \frac{1}{rc^2}\right](l+n) - \frac{\cot\theta}{\sqrt{2}\,r}\,(m - \bar m)\,.
\end{gathered}
\tag{27}
$$

This is our final expression for the spin coefficients. The usual metric case is obtained by setting $c = 1$.

B. Field equations

Now we are in a position to derive and solve the field equations of our theory which are analogs of (vacuum) Einstein equations in GR. The vacuum field equations are obtained by varying action (1) with respect to B:

$$
F + (\Psi + \phi\,\mathrm{Id})B = 0\,. \tag{28}
$$

The components of the curvature of connection (27) are given by

$$
F_- = dA_- + A_- \wedge A_0\,, \quad F_+ = dA_+ + A_0 \wedge A_+\,, \quad F_0 = dA_0 + 2A_- \wedge A_+\,. \tag{29}
$$

Using the explicit spinor form (8) for Ψ, we get

$$
(\Psi + \phi\,\mathrm{Id})B = \tilde X_-(\beta + \phi)B_- + \tilde X_+(\beta + \phi)B_+ + \tilde X(\phi - 2\beta)B_0\,. \tag{30}
$$

Thus, we can write (28) in components:

$$dA_- + A_- \wedge A_0 + (\beta + \phi) B_- = 0\,,$$
$$dA_+ + A_0 \wedge A_+ + (\beta + \phi) B_+ = 0\,, \tag{31}$$
$$dA_0 + 2A_- \wedge A_+ + (\phi - 2\beta) B_0 = 0\,.$$

All computations are straightforward in view of (19) and (27). Each of the first two equations in (31) gives rise to the two differential equations

$$\frac{f_*'}{r g_*^2 f_*} + \left(1 - \frac{1}{c^2}\right) \frac{1}{r^2 g_*^2} = -\frac{g_*'}{r g_*^3} = \beta + \phi\,, \tag{32}$$

while the last equation in (31) gives two additional equations

$$\frac{c^2}{f_* g_*} \left[\frac{f_*}{r g_*} \left(1 - \frac{1}{c^2} + \frac{r f_*'}{f_*}\right)\right]' = \frac{1}{r^2} \left(\frac{1}{g_*^2 - 1}\right) = \phi - 2\beta\,, \tag{33}$$

where

$$f_* := cf\,, \qquad g_* := cg\,. \tag{34}$$

In view of (15) and by virtue of the Bianchi identity $\mathcal{D}F = 0$, where $\mathcal{D}$ is the A-covariant derivative, only three of the four equations (32) and (33) are independent (this can be verified directly), and the system of three independent equations can be presented in the following convenient form:

$$(2 - \phi_\beta)\, \beta' = -\frac{6\beta}{r}\,, \qquad g_*^{-2} = 1 - (2\beta - \phi) r^2\,, \qquad \frac{(f_* g_*)'}{f_* g_*} = \frac{3\phi_\beta}{r(2 - \phi_\beta)}\,. \tag{35}$$

IV. SOLUTION AND ANALYSIS

In this section, we proceed to solving the main system of equations (15), (35), which completely determine the two-form B given by (14) and the one-form A given by (27).

A. Corrections to the metric case

If our solution happens to be in the regime where $|\phi_\beta| \ll 1$, we have $c^2 \approx 1$, which is thus an approximate "metricity" regime. Equations (35) in this approximation lead to the Schwarzschild–(anti)-de Sitter form for the functions f and g in metric (18):

$$\beta = \frac{r_s}{2r^3}\,, \qquad f^2 = g^{-2} = 1 - \frac{r_s}{r} + \phi_0 r^2\,, \tag{36}$$

where ϕ_0 is the value of the almost constant function $\phi(\beta)$ in this domain, and r_s is the Schwarzschild radius, which appears as the integration constant of the solution of the first equation in (35). The constant ϕ_0 corresponds to the effective cosmological constant: $\phi_0 = -\Lambda/3$.

The approximate solution (36) will, in particular, hold in the asymptotic region of large r if ϕ is an analytic function of its arguments $\mathrm{Tr}\,(\Psi^2)$ and $\mathrm{Tr}\,(\Psi^3)$ in the neighborhood of zero. In this case, as can be seen from (6), $\phi(\beta)$ admits an expansion in powers of β^2 and β^3, and the leading term in this expansion for small β is

$$\phi(\beta) = \phi_0 \pm \ell^2\beta^2 + \mathcal{O}\left(\beta^3\right) \,, \tag{37}$$

where ℓ is a constant of dimension length. Thus, we have

$$\phi_\beta \approx \pm 2\ell^2\beta \approx \pm\frac{\ell^2 r_s}{r^3} \tag{38}$$

so that

$$|\phi_\beta| \ll 1 \quad \text{for} \quad r \gg \left(\ell^2 r_s\right)^{1/3} \,. \tag{39}$$

By solving (35) perturbatively in the small parameter $\ell^2 r_s/r^3$, one can obtain the next correction to solution (36) in the domain of large r:

$$\beta = \frac{r_s}{2r^3}\left(1 \pm \frac{\ell^2 r_s}{2r^3}\right) \,, \quad g_*^{-2} = 1 - \frac{r_s}{r}\left(1 \pm \frac{\ell^2 r_s}{4r^3}\right) + \phi_0 r^2 \,, \quad f_*^2 = g_*^{-2}\left(1 \mp \frac{\ell^2 r_s}{r^3}\right) \,. \tag{40}$$

According to (15), the value of the "nonmetricity" parameter c^2 in this approximation is given by

$$c^2 = 1 \mp \frac{3\ell^2 r_s}{2r^3} \,. \tag{41}$$

Thus, we can see that, under the assumption of regularity of the function $\phi\left[\mathrm{Tr}\,(\Psi^2)\,, \mathrm{Tr}\,(\Psi^3)\right]$ in the neighborhood of $\Psi = 0$, the solution in the asymptotic region $r \to \infty$ tends to the metric form (the "nonmetricity" parameter c rapidly tends to zero) asymptotically describing a space of constant curvature. It is in this sense that an analog of the Birkhoff theorem holds in the theory under consideration.

If $\ell \ll r_s$, then the approximate regime (40) and (41) is valid up to the "horizon" $r = r_h$, determined by the condition $f_*^2 = g_*^{-2} = 0$. At the "horizon," the function c remains finite, and the function g_*, hence, also g, diverges. As in the metric case, this can be regarded as a coordinate singularity which can be removed by choosing new time and radial coordinates.

In this way, one can pass to the "black hole" region $r < r_h$, in which the functions f_*^2 and g_*^{-2} change sign and become negative. Somewhere in this region, the condition $|\phi_\beta| \ll 1$ (or, equivalently, $\ell^2 r_s/r^3 \ll 1$) may cease to be valid, and the solution becomes strongly non-metric. To see what can happen in this region, we consider general solution.

B. General solution

First of all, one can note that the first and third equations in (35) are singular at the point where $\phi_\beta = 2$. One can remove this singularity by passing to a new radial coordinate. The value of β itself can be chosen as such a coordinate. Doing this, one can rewrite the system of equations (35) in terms of this new coordinate:

$$\frac{d\log r}{d\beta} = \frac{\phi_\beta - 2}{6\beta}\,, \qquad g_*^{-2} = 1 - (2\beta - \phi) r^2\,, \qquad \frac{d\log(f_* g_*)}{d\beta} = -\frac{\phi_\beta}{2\beta}\,. \tag{42}$$

This system is nonsingular and can easily be integrated. The function c^2, as usually, is given by (15). Note that β has a physical meaning being a scalar characterizing the field Ψ according to (6) and (8).

The metric (18) in the new coordinates (t, β) is written as

$$ds^2 = (1+\phi_\beta)\left[\left(1 - \frac{\phi_\beta}{2}\right)^{-1} f_*^2 dt^2 - \left(1 - \frac{\phi_\beta}{2}\right)\left[\frac{r(\beta)}{3\beta}\right]^2 g_*^2 d\beta^2\right] \\ - r^2(\beta)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \tag{43}$$

where we have used (15) and the first equation of (42).

Metric in this form indicates that the points where $\phi_\beta = 2$, corresponding to $c^2 = 0$, are hypersurfaces across which the coordinates t and β change their space-time roles. At these hypersurfaces, metric (43) is degenerate. One can see that our basic fields remain finite at such points. Indeed, in the new coordinates (t, β), the forms $B_\pm$ are proportional to the functions $cf = f_*$ and $cg = g_*$, which remain finite and nonzero at these points, and the potentially dangerous $l \wedge n$ component in B_0 is also finite:

$$l \wedge n = fg dt \wedge dr = \frac{dr}{d\beta} c^{-2} f_* g_* dt \wedge d\beta = -\frac{r}{3\beta}\left(1 + \phi_\beta\right) f_* g_* dt \wedge d\beta\,, \tag{44}$$

where again we have used (15) and the first equation of (42). The same is true for the spin coefficients $A_\pm$ given by (27), which are proportional to g_*^{-1}. The potentially dangerous part

of the spin coefficient A_0 is finite as well:

$$\frac{1}{\sqrt{2}\,g}\left[\frac{(rcf)'}{rcf}-\frac{1}{rc^2}\right](l+n)=\frac{f_*}{rg_*}\left(\frac{rf_*'}{f_*}+1-\frac{1}{c^2}\right)dt=rf_*g_*(\beta+\phi)dt\,, \tag{45}$$

where we have used equation (32). Due to Eq. (28), the components of the curvature F are also finite.

At a point where $\phi_\beta=-1$, the quantity c^{-2} turns to zero. However, by similar reasoning, one can see that all components of A and B remain finite in the coordinates (t,β). Therefore, the general solution is nonsingular at this point as well. The metric in the form (43) is degenerate, but this is not surprising as we are dealing with an intrinsically nonmetric theory, and the "nonmetricity" parameter c at this point is infinite.

As we already noted in the previous subsection, the condition $g_*^{-2}=0$, hence $f_*^2=0$, corresponds to a horizon in metric (43). This is a coordinate singularity which can be removed by passing to a Kruskal-like coordinate system.

For an illustration, let us consider the function $\phi(\beta)$ exactly in the form

$$\phi(\beta)=\phi_0\pm\ell^2\beta^2\,, \tag{46}$$

with two possible signs. In this case, the solution is

$$r^3(\beta)=\frac{r_s}{2\beta}\,e^{\pm\ell^2\beta}\,,\qquad g_*^{-2}=1-\left(2\beta\mp\ell^2\beta^2-\phi_0\right)r^2(\beta)\,,\qquad f_*g_*=e^{\mp\ell^2\beta}\,, \tag{47}$$

and the function c^2 is given by

$$c^2=\frac{1\mp\ell^2\beta}{1\pm2\ell^2\beta}\,. \tag{48}$$

For $\ell^2\beta\ll1$, this corresponds to the approximate solution (40), (41).

With the upper sign in (46), from the first equation of (47), one can see that there is a minimum value of the radial coordinate

$$r_m^3=\frac{e}{2}\,\ell^2r_s\,, \tag{49}$$

which is precisely the point where $\phi_\beta=2$.

In the case of the lower sign in (46), the function $\beta(r)$ is monotonic, and $\beta\to\infty$ as $r\to0$, which resembles the behavior inside a classical black hole in GR. The point $\phi_\beta=2$ is absent in this case, but we have a singularity in metric (43) at the point $\ell^2\beta=1/2$ corresponding to the condition $\phi_\beta=-1$. As we noted above, the fundamental fields A and B are regular at this point.

C. Minimal black hole

For both signs in (46), there is a minimum value of the "Schwarzschild radius" $r_s = r_*$ for which the black-hole horizon exists. The mechanism, however, is different for the two signs. For the upper sign (corresponding to positive ϕ), there is a minimum value (49) that the coordinate r can take. This value of r is also the minimum of the function g_*^{-2}. For the horizon to exist, this minimum value should be negative. Neglecting, for simplicity, the value of ϕ_0, we can translate it into the condition

$$r_s > \frac{2}{e}\,\ell \tag{50}$$

for the Schwarzschild horizon to exist. If this condition is violated, then, instead of the Schwarzschild-like horizon, one has a naked surface of "non-metricity," which we describe in the following subsection.

For the lower, negative sign in (46), the function g_*^{-2} similarly has a minimum at $\beta = 1/\ell^2$. The condition that this minimum is smaller than zero translates into the condition

$$r_s > \frac{2e}{3^{3/2}}\,\ell \tag{51}$$

for the existence of horizon.

D. Conformal structure

In the modified theory of pure gravity described by action (1), the notion of a unique distinguished metric is replaced by the conformal class of metrics with respect to which the two-form B is self-dual, with metric (18) being a representative of this class. A distinguished physical metric from this class may arise only after one considers gravitational interaction of matter and radiation. Several potential candidates for such a metric can be envisaged at this level. Given the spinor two-form B, one can consider the Urbantke metric $g^{\mathrm{U}}_{\mu\nu}$, defined by the relation [7]

$$\sqrt{|g^{\mathrm{U}}|}\, g^{\mathrm{U}}_{\mu\nu} = \frac{1}{3}\epsilon^{\alpha\beta\gamma\delta}\mathrm{Tr}\left(B_{\mu\alpha}B_{\beta\gamma}B_{\delta\nu}\right) , \tag{52}$$

where the trace is taken with respect to the spinor indices. It is easy to see that the Urbantke metric is related to the metric defined by (18) by the conformal factor $c^{2/3}$.

Another distinguished conformal factor for the metric is obtained by the requirement that the quantity

$$\frac{1}{3}\mathrm{Tr}\,(B \wedge B) \tag{53}$$

coincide with the volume element defined by the new metric. The metric line element that arises this way is given by

$$ds_V^2 = \left(\frac{2c^2+1}{3}\right)^{1/2} ds^2 = (1+\phi_\beta)^{-1/2}\, ds^2\,, \tag{54}$$

where ds^2 is the line element given by (18).

In the metric case, where $c = 1$, all such definitions coincide with our "canonical" metric (18).

As we noted, before one considers coupling of our theory of gravity to matter, it is impossible to distinguish any of the listed possibilities for the metric as being the physical one. However, a rather natural requirement that the coupling of matter degrees of freedom to B is at most quadratic in B (the coupling of YM fields satisfies this requirement) favors the metric defined by (54). Indeed, the usual matter coupling to the Urbantke metric (52) would be non-polynomial in the B field, which is undesirable for many reasons.

In spite of the mentioned ambiguities, the presence of a distinguished conformal class of metrics, with respect to which the two-form field B is self-dual, allows us to speak about the conformal structure of the obtained solution. This conformal structure can be shown to have physical meaning reflecting the geometry of propagation of light.

The conformal structure of our black-hole solution in the (t, β) coordinate plane will strongly depend on the shape of our basic function $\phi(\beta)$. However, its key details are easy to understand by looking at the form of metric (43) in the (t, β) coordinates. The metric described by (43) has the following types of critical surfaces defined by special positions in the β coordinate:

1. The point where $g_*^{-2} = 0$, hence, also $f_*^2 = 0$ [the product $f_* g_*$ is positive and finite for finite β in view of the last equation in (42)]. It defines a null horizon, analogous to the Schwarzschild horizon, separating different space-time regions in the black-hole solution. This is a coordinate singularity in (t, r) or (t, β) coordinates, which can be removed by proceeding to Kruskal-like coordinates.

2. The point where $\phi_\beta = 2$, or $c^2 = 0$. By differentiating the function g_*^{-2}, one can verify that this is a critical point (maximum or minimum) of this function, as well as of the radius $r(\beta)$ as a function of β. Hence, typically, this critical point will not coincide with the previous one, where g_*^{-2} vanishes. This is a position of true singularity in the class of metrics (43). However, as we said before, the solution in our basic fields A and B can well be extended beyond this surface. What happens at this point in terms of our basic two-form field B is that the components $B_\pm$ and B_0 no longer span a subspace in the space of two-forms which is self-dual in any metric. One can notice, for example, that $cgdr = g_* \, (dr/d\beta) \, d\beta = 0$ at this point, so the one-forms

$$cn = cl = \frac{1}{\sqrt{2}} f_* dt \,, \tag{55}$$

and the anti-self-dual two-forms

$$\tilde{B}_- = c\, m \wedge n \,, \qquad \tilde{B}_+ = c\, l \wedge \bar{m} \,, \tag{56}$$

coincide with their self-dual counterparts B_- and B_+, respectively. This means precisely that there is no metric with respect to which the tripple $B_\pm, B$ is self-dual.

3. The point where $\phi_\beta = -1$, or $c^{-2} = 0$. This is another singularity in metric (43). At this point, we have $l \wedge n = 0$ in view of (44), and the two-form B_0 coincides with its anti-self-dual counterpart $\tilde{B}_0$:

$$B_0 = l \wedge n - m \wedge \bar{m} = -\,(l \wedge n + m \wedge \bar{m}) = \tilde{B}_0 \,. \tag{57}$$

Again, the fields A and B are well-behaved at this point, so we can cross it and proceed to a neighboring space-time region.

As an example, in Fig. 1 we have pictured the conformal diagram obtained for the function $\phi(\beta) = \ell^2 \beta^2$, which has only critical surfaces of type 1 and 2. The Schwarzschild-like horizons are pictured by solid lines, while the dashed lines correspond to horizons of type 2. The radius r has an absolute minimum value, in this case given by (49), which is reached precisely at horizons of type 2. As $\beta \to \infty$, which corresponds to $r \to \infty$, one approaches a “singularity” indicated by thick dashed lines.

Since, in the absence of matter couplings, the physical metric is not specified, the issue of “geodesic completeness” of solution is not well defined in the purely gravitational theory

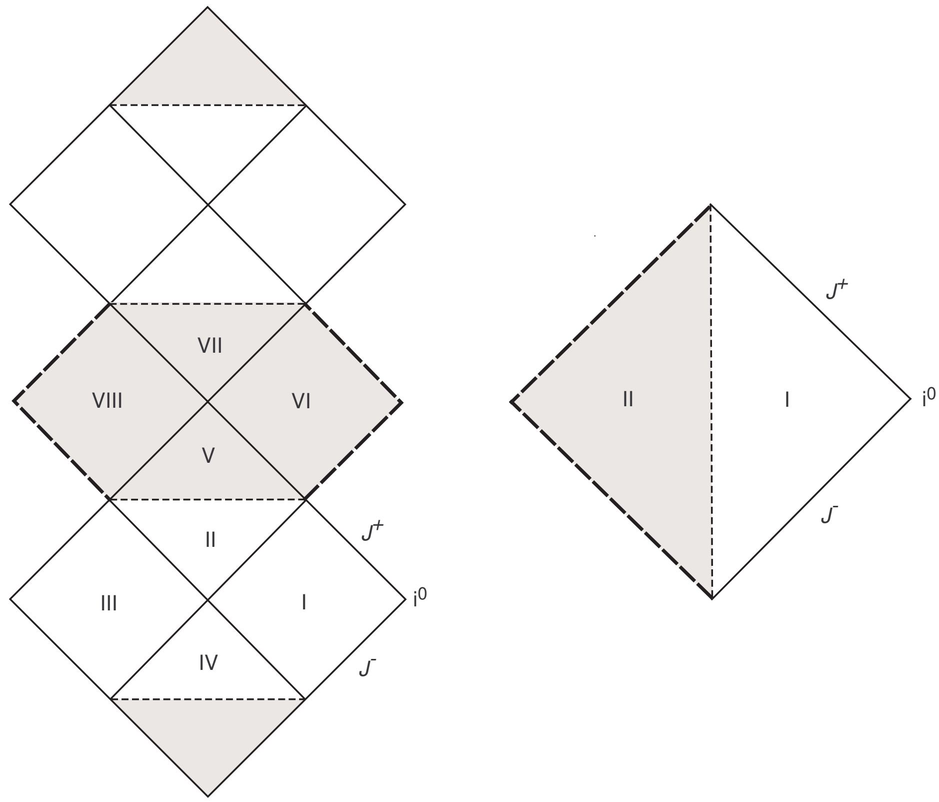

FIG. 1: Conformal diagram of the spherically symmetric solution with the function $\phi(\beta) = \ell^2\beta^2$ and with the Schwarzschild radius satisfying (left image) and violating (right image) condition (50). Different regions are numbered in such a way that the coordinate t is timelike in the odd regions, and spacelike in the even ones. Solid lines between regions indicate Schwarzschild-like horizons of type 1, at which $g_*^{-2} = 0$. Thin dashed lines indicate singular surfaces of type 2, where metric ceases to exist. The flow of time is vertical in the white regions and changes to horizontal in the grey regions. Thick dashed lines indicate the true singularity, where $r = 0$ and $\beta = \infty$. The configuration on the left image extends periodically and indefinitely upward and downward. We are living in one of the regions of type I; the asymptotic spatial infinity in this region is denoted by i^0, and the future and past null infinities are denoted by $\mathcal{J}^+$ and $\mathcal{J}^-$, respectively.

under consideration. Nevertheless, it is interesting and instructive to see whether the metric defined by (18) is geodesically complete as one approaches the singularity $\beta \to \infty$ in the regions of type VI and VIII. It is easy to see that null geodesic completeness in such regions

is equivalent to the divergence of the integral

$$\int^{\infty} \frac{1+\phi_\beta}{\beta} f_* g_* r(\beta) d\beta\,. \tag{58}$$

Thus, solution (47), (48) with the upper sign has convergent integral (58), hence, the metric (18) defined by this solution is not null geodesically complete. However, it is timelike geodesically complete. Indeed, in the regions in question, β plays the role of the time coordinate since $c^2 \to -1/2$ as $\beta \to \infty$. To the leading order $g_*^{-2} \sim \ell^2\beta^2 r^2(\beta)$ and $dr \sim (\ell^2/3) r(\beta) d\beta$. This gives $g^2 dr^2 \sim -2\ell^2 d\beta^2/9\beta^2$, and the proper time to infinity $\beta \to \infty$ is logarithmically divergent.

For the lower sign in (47), (48), integral (58) is divergent; therefore, the corresponding metric (18) is null geodesically complete in the region $\beta \to \infty$, or $r \to 0$. However, it is not timelike geodesically complete since the timelike distance to the singularity $\int g dr \propto \int dr$ is finite (the coordinate r is timelike and $r \to 0$ in this case).

The described behavior makes the "singularity" at $\beta \to \infty$ rather an interesting place. We have depicted it as a null surface because, in some respects, it is reminiscent of the usual null infinity. The reader should, however, keep in mind that the detailed structure of this "singularity" is quite unusual, and strongly depends on the details of the behavior of the function ϕ as $\beta \to \infty$. In contrast, the structure of the "non-metric horizons" at $\phi_\beta = 2, -1$ is universal. It is also worth emphasizing that the singularity at $\beta \to \infty$ is located "outside" of the black hole on the left image in Fig. 1 in the sense that another Schwarzschild-like horizon has to be crossed to reach it. It is in this sense that the theory under consideration "resolves" the singularity inside the black hole.

In the examples considered above, we have assumed that the function $\phi(\beta)$ behaves as $\pm\ell^2\beta^2$ as $\beta \to \infty$. Some other choices are possible and may be interesting to consider. Thus, it is easy to devise $\phi(\beta)$ so that there is a maximal possible "curvature" in the theory. To achieve this, one can choose $\phi(\beta)$ diverging for some finite value of β. A good example of such a function is given by $\phi(\beta) = x^2/\ell^2(1-x^2)$, where, as before, $x = \ell^2\beta$. It diverges at the value $\beta_{max} = 1/\ell^2$, which plays the role of maximal curvature in the theory. The qualitative behavior of the conformal diagram for the spherically symmetric solution will be the same in this case. The details of the behavior at the singularity at β_{max} will, however, be different. Another interesting possibility not considered in this paper is that, for Planckian curvatures, the function $\phi(\beta)$ changes rapidly — so that, in particular, one of the non-metric

horizons at $\phi_\beta = 2, -1$ is reached — and approaches a large constant value for β much larger than Planckian. In this scenario, we would get a conformal diagram consisting of elements similar to the ones described above, with a metric universe with large cosmological constant on the other side of the black hole. It could be interesting to study all such possibilities in more detail.

V. MODIFICATION OF GRAVITY AT DIFFERENT CURVATURES

The theory under consideration leads to a possibility of scale-dependent modification of gravity of very interesting nature.

Defining the *effective* Schwarzschild radius $r_s(\beta)$ by the relation

$$r_s(\beta) = 2\beta r^3(\beta)\,, \tag{59}$$

we can present our main system of equations (42) in the form

$$\frac{d\log r_s}{d\beta} = \frac{\phi_\beta}{2\beta}\,, \qquad g_*^{-2} = 1 - \frac{r_s}{r} + \phi r^2\,, \qquad \frac{d\log(f_* g_*)}{d\beta} = -\frac{\phi_\beta}{2\beta}\,. \tag{60}$$

Integrating the first and the last equations in (60), we get the following relations, valid for arbitrary β_1 and β_2:

$$\frac{r_s(\beta_2)}{r_s(\beta_1)} = Z(\beta_1, \beta_2)\,, \qquad \frac{f_*(\beta_2)}{f_*(\beta_1)} = \frac{g_*(\beta_1)}{g_*(\beta_2)}\, Z^{-1}(\beta_1, \beta_2)\,, \tag{61}$$

where

$$Z(\beta_1, \beta_2) = \exp\left(\int_{\beta_1}^{\beta_2} \frac{\phi_\beta}{2\beta} d\beta\right)\,. \tag{62}$$

The first equation in (61) shows that we are dealing with a theory in which the effective Schwarzschild radius becomes distance-dependent, and the second equation indicates the presence of an additional redshift/blueshift factor in the metric.

Assume that there exist two domains in the space of values of β, in both of which one has $|\phi_\beta| \ll 1$, hence, both of which are characterized by the condition of metricity. Then the Schwarzschild radii will be constant in the corresponding regions of the radial coordinate and will be related by (61) in which β_1 and β_2 are the representative values of β in the corresponding domains. The time-time component of the metric, according to the second relation of (61), will exhibit the relative redshift factor between these two regions

$$\frac{g_{00}^{(2)}}{g_{00}^{(1)}} = \frac{g_{rr}^{(1)}}{g_{rr}^{(2)}}\, Z^{-2}\,. \tag{63}$$

The cosmological constant, in general, will also be renormalized.

As an illustration for these effects, we consider an explicit example, which is defined by the function

$$\phi(\beta) = \phi_0 - \phi_1 \log\left(1 + x^2\right) , \qquad x := \ell^2 \beta . \tag{64}$$

The derivative of ϕ is given by the expression

$$\phi_\beta = -\frac{2\alpha x}{1 + x^2} , \tag{65}$$

where

$$\alpha = \ell^2 \phi_1 \tag{66}$$

is a dimensionless parameter. Function (64) has the property that $|\phi_\beta| \ll 1$ in the two asymptotic regions $x \ll (1 + |\alpha|)^{-1}$ and $x \gg 1 + |\alpha|$, so that the theory is approximately "metric" in both these regions.

Equations (42) can be integrated in this case, with the solution

$$\frac{\ell^2 r_s}{2r^3} = x e^{\alpha \arctan x - \frac{\pi\alpha}{2}} , \qquad f_* g_* = e^{\alpha \arctan x - \frac{\pi\alpha}{2}} . \tag{67}$$

We have chosen the integration constants in (67) so as to obtain the standard Schwarzschild solution at small radial distances, $r^3 \ll (1 + |\alpha|)^{-1} \ell^2 r_s$, where $x \gg (1 + |\alpha|)$. At these distances, we can expand the right-hand sides of solution (67) to obtain

$$\frac{\ell^2 r_s}{2r^3} = x \left[1 - \frac{\alpha}{x} + \mathcal{O}\left(\alpha x^{-3}\right)\right] , \qquad f_* g_* = 1 - \frac{\alpha}{x} + \mathcal{O}\left(\alpha x^{-3}\right) , \quad x \gg (1 + |\alpha|) . \tag{68}$$

We also have

$$c^2 = 1 + \frac{3\alpha}{x} + \mathcal{O}\left(\alpha x^{-2}\right) , \quad x \gg (1 + |\alpha|) . \tag{69}$$

Collecting all the terms, we obtain the leading contribution to the g_{00} coefficient in metric (18) and $\tilde{g}_{00}$ coefficient in the Urbantke and "volume" metric (which coincide in this approximation):

$$\begin{aligned} g_{00} &= 1 - \frac{r_s}{r} + \left[\phi_0 - 2\phi_1 \left(1 + \log \frac{\ell^2 r_s}{2r^3}\right)\right] r^2 - \frac{10\alpha r^3}{\ell^2 r_s} , \\ \tilde{g}_{00} &= 1 - \frac{r_s}{r} + \left[\phi_0 - 2\phi_1 \left(1 + \log \frac{\ell^2 r_s}{2r^3}\right)\right] r^2 - \frac{8\alpha r^3}{\ell^2 r_s} , \quad r^3 \ll \frac{\ell^2 r_s}{1 + |\alpha|} . \end{aligned} \tag{70}$$

At large radial distances $r^3 \gg (1 + |\alpha|) \ell^2 r_s$, where $x \ll (1 + |\alpha|)^{-1}$, we have

$$\frac{\ell^2 r_s}{2r^3} = e^{-\frac{\pi\alpha}{2}} x \left[1 + \alpha x + \mathcal{O}\left(\alpha x^3\right)\right] , \qquad f_* g_* = e^{-\frac{\pi\alpha}{2}} \left[1 + \alpha x + \mathcal{O}\left(\alpha x^3\right)\right] , \tag{71}$$

$$c^2 = 1 + 3\alpha x + \mathcal{O}\left(\alpha x^2\right) \,, \quad x \ll \frac{1}{1+|\alpha|} \,. \tag{72}$$

In this case, we obtain the following leading contribution to the corresponding metrics, up to terms of order $\ell^2 r_s/r^3$:

$$\begin{aligned} g_{00} &= e^{-\pi\alpha}\left[1 - e^{\frac{\pi\alpha}{2}}\frac{r_s}{r} + \phi_0 r^2 - e^{\frac{\pi\alpha}{2}}\frac{\alpha\ell^2 r_s}{2r^3}\right] \,, \\ \tilde{g}_{00} &= e^{-\pi\alpha}\left[1 - e^{\frac{\pi\alpha}{2}}\frac{r_s}{r} + \phi_0 r^2\right] \,, \quad r^3 \gg (1+|\alpha|)\,\ell^2 r_s \,. \end{aligned} \tag{73}$$

We note that $c^2 \approx 1$ in our approximation in these regions, so the physical metric is well defined. In fact, the contributions proportional to α in (70) and (73) are small and can be dropped if α is not very large.

In the intermediate region of radial distances

$$\frac{1}{1+|\alpha|} \lesssim \frac{r^3}{\ell^2 r_s} \lesssim 1 + |\alpha| \,, \tag{74}$$

the metric coefficient behaves in a complicated way; moreover, the physical metric is not well defined in this region unless $\alpha \ll 1$. However, even without the detailed knowledge of matter couplings to gravity, simply on the basis of physical continuity, it is obvious that the *apparent* mass (or gravitational coupling) will be varying continuously between the two asymptotic regions. The intermediate region (74) is rather extended if $|\alpha| \gg 1$.

It is interesting to note that the shift in the cosmological constant between the two asymptotic regions will be absent in a special case where the function $\phi(\beta)$ takes the same value in the corresponding domains of the β space. In spite of the absence of such a shift, the Schwarzschild radius will still get renormalized. This can be illustrated by another example:

$$\phi(\beta) = \phi_0 - \phi_1 \frac{x^2}{1+x^3} \,, \qquad x := \ell^2\beta \,, \tag{75}$$

which is a function of both invariants $\mathrm{Tr}\,(\Psi^2) = 6\beta^2$ and $\mathrm{Tr}\,(\Psi^3) = -6\beta^3$ in the case $\alpha = 0$ [see Eq. (6)]. This function is characterized by the property that $\phi \approx \phi_0$ in both asymptotic domains $x \ll 1$ and $x \gg 1$. The mass renormalization exponent in (B9) is given in this case by the integral

$$-\int_0^\infty \frac{\phi_\beta}{2\beta}\, d\beta = \alpha \int\limits_0^\infty \frac{1 - \frac{1}{2}x^3}{\left(1+x^3\right)^2}\, dx = \frac{\pi\alpha}{3\sqrt{3}} \,, \tag{76}$$

where $\alpha = \ell^2\phi_1$.

In the case under consideration, we obtain the following approximate expressions for the metric, respectively, at small and large radial distances:

$$g_{00} = 1 - \frac{r_s}{r} + \phi_0 r^2 + 2\alpha \left(\frac{2r^3}{\ell^2 r_s}\right)^2 ,$$
$$\tilde{g}_{00} = f^2 = 1 - \frac{r_s}{r} + \phi_0 r^2 + \frac{3}{2}\alpha \left(\frac{2r^3}{\ell^2 r_s}\right)^2 , \qquad r^3 \ll \frac{\ell^2 r_s}{1+|\alpha|} , \tag{77}$$

$$g_{00} = e^{-\frac{2\pi\alpha}{3\sqrt{3}}} \left[1 - e^{\frac{\pi\alpha}{3\sqrt{3}}} \frac{r_s}{r} + \phi_0 r^2 - e^{\frac{\pi\alpha}{3\sqrt{3}}} \frac{\alpha \ell^2 r_s}{2r^3}\right] ,$$
$$\tilde{g}_{00} = e^{-\frac{2\pi\alpha}{3\sqrt{3}}} \left[1 - e^{\frac{\pi\alpha}{3\sqrt{3}}} \frac{r_s}{r} + \phi_0 r^2\right] , \qquad r^3 \gg (1+|\alpha|)\, \ell^2 r_s . \tag{78}$$

More general set of examples, characterized by arbitrary powers in the region of large and small curvatures is considered in Appendix B. The above formulas are in correspondence with the generic expressions (B4) and (B8).

VI. PHYSICAL EFFECTS

A. Modified gravity instead of dark matter

Modifications of gravity at large distances are currently under consideration as an alternative to the dark-matter phenomenon (see, e.g., [8]). Our theory can be a viable candidate in this respect. Indeed, we have seen in the previous section that the gravitational strength of the central spherically symmetric body is distance-dependent. Consider, for definiteness, our example (64). Because the parameter α stands in the exponent of the renormalized gravitational mass [see Eq. (73)], one should have positive $\alpha \sim 1$ in order that the gravitational mass increase with distance. Then the fundamental length scale ℓ should be chosen in such a way as to ensure that the deviations from the Newtonian behavior begin at a certain distance from the gravitating body. A typical representative of such a situation is a spiral galaxy like our Milky Way, of mass $M_g \sim 10^{11} M_\odot$, in which deviations from Newton's behavior (flat rotation curves) are prominent at a distance $r_g \sim 10$ kpc. This gives us the estimate

$$\ell \sim \sqrt{\frac{r_g^3}{r_s}} \sim 10 \text{ Mpc} , \tag{79}$$

where $r_s = 2GM_g \sim 10^{-2}$ pc is the Schwarzschild radius associated with the galaxy mass contained inside the radius r_g. Interestingly, this estimate for ℓ roughly corresponds to the

scale on which the relative perturbation $\delta\rho/\rho$ of the density in the universe becomes of order unity today.

The critical distances $r_c \sim (\ell^2 r_s)^{1/3}$ for the Sun, for the Earth, and for an isolated proton are then given by

$$r_\odot \sim 2 \text{ pc}\,, \qquad r_\oplus \sim 2\times 10^3 \text{ au}\,, \qquad r_p \sim 5 \text{ mm}\,, \tag{80}$$

respectively. The large practical value of the critical radius (80) even for microscopic bodies such as a proton will result in the effect that, in most (if not all) practical cases in the given example, the overall redshift factor (63) of low-curvature regions will be unobservable. In other words, the photons (and other particles) will always be emitted from regions of relatively high curvature (magnitudes of curvature corresponding to $\ell^2\beta \gg 1$ in the spherically symmetric case) and observed also in regions of high curvature, and there is no relative redshift factor between such regions.

To explain this in more detail, assuming that we have N gravitational sources separated by distances large compared to their critical scales $r_c^{(i)} = \left(\ell^2 r_s^{(i)}\right)^{1/3}$ determined by their small-distance Schwarzschild radii $r_s^{(i)}$, $i = 1, \ldots, N$, we can write the metric in the region far from all such sources as

$$g_{00} = 1 - \sum_i \frac{\tilde{r}_s^{(i)}}{r_i}\,, \qquad r_i \gg r_c^{(i)}\,, \quad i = 1, \ldots, N\,, \tag{81}$$

where r_i is the distance from the point of observation to the i^{th} source, and $\tilde{r}_s^{(i)} = Z r_s^{(i)}$ is the renormalized Schwarzschild radius with the universal factor Z. We have used the superposition principle clearly valid in the domain $\ell^2\beta \ll 1$ and neglected the contribution from the effective cosmological constant. As we approach any of the sources within its critical radius, the metric is dominated by this source and reads

$$g_{00}^{(i)} = Z^2 \left(1 - \frac{r_s^{(i)}}{r_i}\right)\,, \quad r_i \ll r_c^{(i)}\,, \tag{82}$$

with the same universal redshift factor Z. Because of this universality, a photon or any other particle traveling from one such source to another one will experience no additional relative redshift.

Thus, for any sufficiently massive practical observer, only the effect of the apparent increase of the central mass (or, equivalently, of the gravitational constant) with distance

will be detectable, which asymptotically will result in the difference between r_s and $\tilde{r}_s = Zr_s$. Although the notion of physical metric may not be well defined at intermediate distances (74), it is clear that the effect of apparent increase of the gravitational mass of the central body should be continuous with distance; hence, the value of the effective mass in our example will monotonically interpolate between its asymptotic magnitudes r_s and Zr_s as one moves away from the gravitating body. This potentially can be used to explain the effect of missing gravitating mass in the universe.

B. Pioneer anomaly

It is interesting to see whether the theory under consideration might be able to explain the observed anomalous acceleration of the Pioneer spacecraft $a_{\rm P} \simeq 8 \times 10^{-10}$ m/s^2 [9]. To explain the alleged missing mass in galaxies one has to take the function $\phi(\beta)$ such that the effective gravitating mass increases with the distance. This would imply the presence of anomalous acceleration directed towards the Sun, potentially matching the observed value.

To derive the exact observable result, we need to consider concrete realistic functions $\phi(\beta)$ as well as the couplings of matter to the gravitational degrees of freedom of this theory, which will possibly distinguish one of the effective metrics from the whole conformal class. Some preliminary order-of-magnitude estimates can already be made irrespective of such details. The leading terms in non-metric corrections to the acceleration at small distances are expected to be of the order

$$a_0 \simeq \frac{\alpha r^2}{\ell^2 r_s}\,, \tag{83}$$

as can be seen from the last terms on the right-hand side of (B4) or (B5) of Appendix B for the simplest generic case $p_2 = 1$ [see also the concrete example (70)]. (To avoid confusion, we remind the reader that the speed of light was set to unity throughout this paper.) For the value of $\ell/\sqrt{\alpha} \simeq 20$ Mpc, the Pioneer acceleration $a_{\rm P} \simeq 8 \times 10^{-10}$ m/s^2 will be reached at a distance from the Sun $r = 20$ au. This value of $\ell/\sqrt{\alpha}$ is of the same order as that which we obtained in the explanation of the missing mass in galaxies [see Eq. (79)]. Thus, together with accounting for the missing mass in galaxies, the present theory may also be able to explain this observed anomaly in a uniform way.

Because of the quadratic dependence on the distance in (83), the anomalous acceleration of the inner planets of the solar system will be well below the observational limits. However,

Object	Distance (au)	$a_{\rm P}^{\rm theor}$ $\left(10^{-10}\,{\rm m/s^2}\right)$	$a_{\rm P}^{\rm obs}$ $\left(10^{-10}\,{\rm m/s^2}\right)$
Mercury	0.39	0.003	0.04
Icarus	1.08	0.02	6.3
Mars	1.52	0.04	0.1
Jupiter	5.2	0.5	0.12
Uranus	19.2	7	0.08*
Neptune	30.1	17	0.13*

TABLE I: The final column is the upper limit on constant acceleration determined from planetary orbits and taken from [10]. The constraints imposed by the orbits of Uranus and Neptune are somewhat uncertain; for this reason, they are marked with asterisk (see [10]). The third column is the theoretical anomalous acceleration estimated from (83) with $\ell/\sqrt{\alpha} = 20$ Mpc. Theoretical estimates for Uranus and Neptune, if proved to be valid, significantly exceed the observational constraints, which are also inconsistent with the Pioneer anomaly as a modification of gravity [10].

the theoretical prediction for the outer planets based on the simple estimate (83) seems to disagree with the observational constraints, as can be seen from Table I. The observational constraints for Uranus and Neptune, if proved to be valid, are also inconsistent with the general explanation of the Pioneer anomaly as a modification of gravity [10].

The anomalous acceleration (83) is obtained for functions $\phi(\beta)$ which have logarithmic asymptotic behavior at relatively large values of β, as in example (64), which results in metric (70) at small distances. If the function $\phi(\beta)$ has power-law behavior at large values of β, as in example (75), then the modified metric at small distances takes the form (77), leading to the anomalous acceleration of the order

$$a_0 \simeq \frac{\alpha}{r} \left(\frac{r^3}{\ell^2 r_s} \right)^2 , \tag{84}$$

which differs from (83) by a small factor $r^3/\ell^2 r_s$, of the order 10^{-14} for Uranus and Neptune. In this case, the expected anomalous acceleration is many orders of magnitude smaller than that observed for the Pioneer, and satisfies well the solar-system constraints derived from planetary motion.

Thus we can see that the specific prediction for anomalous accelerations within the solar system is strongly sensitive to the form of the unknown function $\phi(\beta)$ in the domain of

relevant curvatures β. It looks possible to explain the Pioneer anomaly (if it has gravitational origin) in frames of the modified theory of gravity under consideration, but a concrete realization of this requires more work.

C. Non-cosmological redshifts of quasars and gamma-ray bursts

The idea that high redshifts of quasars are not of cosmological nature, but rather are caused by some poorly understood physical circumstances, is pursued by a number of astrophysicists [11] (see also [12] for critical assessment and [13] for a historical review). Similar considerations exist concerning the high redshifts of gamma-ray bursts [14]. One of the problems with this controversial hypothesis is to propose a viable alternative explanation for such high redshifts. Here we would like to demonstrate that our theory may be capable of providing such an explanation.

Suppose that the condition $\phi_\beta > 0$ is satisfied in the range $\beta_1 < \beta < \beta_2$ so that the redshift factor $Z(\beta_1, \beta_2)$ in (62) is bigger than unity. Then, assuming for the moment that the end points of this range are in the metricity regime $\phi_\beta \ll 1$, and taking into account that the value of β in the corresponding regions is the value of the Weyl curvature of the physical metric, we conclude that light emitted from regions of higher Weyl curvature will be additionally redshifted by the factor $Z(\beta_1, \beta_2) > 1$. The observable redshift $z_{\rm obs}$ of a source with such intrinsic redshift factor Z and with local (peculiar or cosmological) redshift z will then be given by

$$z_{\rm obs} = Z(1+z) - 1\,. \tag{85}$$

This effect can be responsible for the observed high redshifts of quasars and gamma-ray bursts.

The value of $Z(\beta_1, \beta_2)$ is quite *arbitrary* depending exponentially on the behavior of the function $\phi(\beta)$. As a simple example, consider the function

$$\phi(\beta) = \phi_1 \log\left(1 + x^2\right)\,, \qquad x = \ell^2 \beta\,, \tag{86}$$

which gives the "metric" behavior in the two domains $x \ll 1$ and $x \gg 1$. The derivative ϕ_β should also satisfy the condition $\phi_\beta < 2$ in order that we do not encounter a singularity outside the horizon. The validity of this condition for all β leads to the constraint

$$\alpha := \frac{\ell^2 \phi_1}{2} < 1\,. \tag{87}$$

The redshift factor is given by the expression

$$Z = \exp\left(\int\limits_{x_1 \ll 1}^{x_2 \gg 1} \frac{\phi_\beta(x)}{2x} dx \right) \approx e^{\pi\alpha} < 23.14 \,, \tag{88}$$

where the last upper bound comes from the bound in (87). The upper bound (87), (88) for the admissible values with function (86) is more than enough to explain the redshifts of quasars and gamma-ray bursts up to $z_{\rm obs} \lesssim 6$. Indeed, we need $\alpha \approx 0.62$ to get the highest values of $z_{\rm obs} \approx 6$ for $z = 0$.

According to the described scenario, additional redshift factor Z will be present in *all* massive compact objects in which radiation originates in regions with curvature $\beta \gtrsim \ell^{-2}$, i.e., having the following relation between the size R and Schwarzschild radius r_s :

$$R^3 \lesssim r_s \ell^2 \,. \tag{89}$$

The free parameters ℓ and α should be fitted to model redshifts of compact objects with strong gravity. Thus, if we take an accretion onto a black hole as a working model for a quasar, then we will have $R \sim r_s$, and condition (89) will become $r_s \lesssim \ell$, implying that only black holes of mass *smaller* than some value produce redshifted emission. To fix the value of ℓ, one needs to build a detailed model of a quasar, which obviously goes beyond the scope of this paper. We only stress that now it has to be built under the assumption that quasars are not superluminous objects situated at very large distances but rather are compact massive objects of average absolute luminosity in our proximity. Also note that the gravitational redshift in our theory is different by nature to the usual gravitational redshift from a massive body in GR. This is clear from the observation that our effect involves a length scale ℓ in (89) and, therefore, can take place at arbitrary Weyl curvatures, for example, at which the usual redshift would be negligible. The redshift itself depends on another free constant α, as can be seen from (88). Moreover, our effect in a wide range of curvature values around the central body can be almost independent of the value of curvature itself. Hence, the usual constraints [15] that rule out the gravitational redshift of quasars in general relativity have to be revised in the theory under investigation.

There are two possible sources of the observed scatter in the values of redshifts. The redshift factor $Z(\beta_1, \beta_2)$ will depend on the actual curvature β_2 of the region where radiation is formed. We remember that the effective physical metric is not yet defined in the region

where $|\phi_\beta| \sim 1$; however, the observable effect of redshift from these regions is expected to be of the magnitude $Z(\beta_1, \beta_2)$. The spread in the values of β_2 will then lead to a spread in the observed redshifts. Another obvious source of scatter in the redshift values is the overall cosmological expansion and peculiar motion which contribute to the observable redshift (85). As is clear from (85), the effect due to peculiar motion and cosmological expansion is *amplified* by the factor Z. For example, if $Z = 2$, then a nearby quasar situated at $z = 0$ will be observed at $z_{\rm obs} = 1$, while a quasar situated at $z = 0.5$ will be seen to have $z_{\rm obs} = 2$.

The non-cosmological redshift effect which we consider in this subsection could also be relevant to neutron stars. In order that this effect be appreciable, condition (89) should be satisfied at their surfaces. Considering that neutron stars have approximately nuclear density, we conclude that the ratio R^3/r_s (where R is their radius) is approximately constant in such objects, and condition (89) then gives a numerical estimate for ℓ:

$$\ell \gtrsim 30 \text{ km} . \tag{90}$$

With ℓ satisfying this condition, radiation from the surfaces of neutron stars will be additionally effectively redshifted. This effect, in principle, may be tested by observations.

Concluding this subsection, we note that the same condition (90) will formally imply nontrivial gravitational effects from the region of atomic nuclei. However, the classical picture of gravity that we considered here may be not valid in such microscopic regions. This question deserves additional investigation.

D. Combined scenario

In order to implement simultaneously the described physical effects, the function $\phi(\beta)$ much be chosen appropriately. It is clear that we require the existence of three different domains in the β space, with solar system in the region characterized by the "metricity" property $\phi_\beta \ll 1$. Let us denote representative values of β in the order of increasing by β_1, β_2, and β_3. The solar-system values of curvature will correspond to β_2. We should have $Z(\beta_1, \beta_2) < 1$ to explain galactic rotation curves, and $Z(\beta_2, \beta_3) > 1$ to explain high redshifts of quasars and gamma-ray bursts. The function that may do the job will look something like

$$\phi(\beta) = \phi_0 - \phi_1 \log\left(1 + x_1^2\right) + \phi_2 \log\left(1 + x_2^2\right) , \qquad x_1 = \ell_1^2 \beta , \quad x_2 = \ell_2^2 \beta , \tag{91}$$

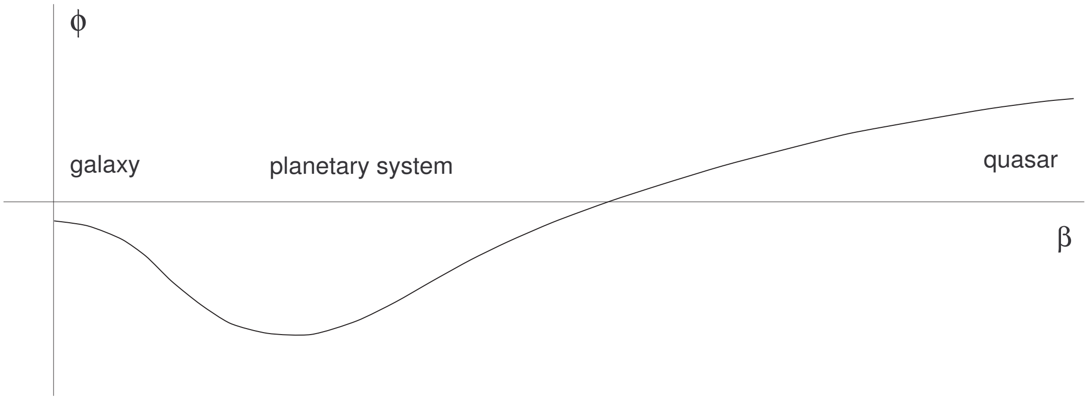

FIG. 2: Typical qualitative shape of the function $\phi(\beta)$ which might exhibit the three effects discussed in this paper, namely, missing mass in galaxies, anomalous acceleration in the outskirts of the solar system, and high redshifts of quasars and gamma-ray bursts. The titles of astrophysical objects are placed at the corresponding values of β, which is roughly the Weyl curvature of space-time characteristic of these objects.

with $\ell_1 \gg \ell_2$ and $\phi_2 > \phi_1$. The value of ϕ_0 should be negative to account for the positive large-scale cosmological constant $\Lambda = -3\phi_0$. Qualitative behavior of the function $\phi(\beta)$ is shown in Fig. 2. However, one can propose other functions with the desired properties, and it is by combined theoretical investigations in quantum gravity and by comparison with observations that a reasonable function $\phi(\beta)$ may be established.

VII. SUMMARY

In this paper, we considered the vacuum spherically symmetric problem in the theory of gravity described by action (1), which modifies the Plebański self-dual formulation of general relativity by introducing an additional function $\phi\left[\mathrm{Tr}\left(\Psi^2\right), \left(\Psi^3\right)\right]$. In the spherically symmetric case, the two arguments of the function ϕ are not independent, and it becomes a function of only one effective field β, defined in (6). The form of the function ϕ is to be determined from quantum considerations. Since this problem is not yet solved, in concrete physical applications we have explored several different possibilities of its form.

The basic variables of the theory under consideration are the spin connection one-form A and the two-form spinor field B, while the Lagrange multiplier field Ψ can be thought

of as expressible in terms of B via the algebraic "metricity" equations. The theory with nontrivial $\phi(\beta)$ does not distinguish any special metric; however, there arises a conformal class of metrics with respect to which the two-form B is self-dual. The existence of such a conformal class enables us to speak about the conformal structure of the solution.

The vacuum spherically symmetric solution has several significant features, and we can summarize our results as follows:

1. For arbitrary ϕ, the theory respects the analog of the Birkhoff theorem, i.e., the vacuum spherically symmetric solution is necessarily static and asymptotically describes a space of constat curvature. In our theory, this means the existence of a vector field generating a group of diffeomorphisms that leave invariant the fields A, B and Ψ. In the coordinate language, the components of these fields do not depend on the "time" coordinate t. This property is probably connected with the fact that our modifications of the Plebański formulation of the general-relativistic action does not increase the order of differential equations. A similar property was recently proved for metric theories which preserve the second-order character of field equations [16].

2. In any domain of its argument in which the function $\phi(\beta)$ varies slowly ($\phi_\beta \ll 1$), it acts simply as a (multiple of) the cosmological constant, and the solution in the corresponding spatial domain possesses approximate "metricity" property in the sense that the three components of the self-dual spinor two-form B are expressible as self-dual parts of the canonical exterior products of some basis one-forms. These basis one-forms then define a unique metric in the corresponding space-time region, which is an approximate solution of the vacuum Einstein equations. In our spherically symmetric solution, such a behavior is obtained at spatial infinity, where we recover the Schwarzschild metric with small and rapidly decaying "nonmetric" corrections. The value of β in this case plays the role of Weyl curvature.

3. If several regions exist in the domain of β where the function $\phi(\beta)$ is slowly varying, then the distinguished metric in the corresponding spatial regions is described by the Schwarzschild-de Sitter form but with different values of the Schwarzschild radius and effective cosmological constant. In other words, the gravitational and cosmological constants become curvature-dependent in this theory. This property of the solution can potentially be used to account for the problem of missing mass in spiral galaxies and other astrophysical objects. The nonmetric corrections in the regions close to the gravitating bodies in this case

could also explain the observed anomaly in the acceleration of the Pioneer spacecraft.

4. In addition to this effective remormalization of gravitational and cosmological constants, there arises a nontrivial universal redshift factor between regions of different Weyl curvature β. This effect potentially can explain the high redshift of quasars and gamma-ray bursts.

5. The conformal structure of our solution inside the black-hole region is different from that of the Schwarzschild solution and depends on the form of the function ϕ. Typically, "inside" the analog of the Schwarzschild horizon one finds another surface of extreme "non-metricity". This surface is spacelike, and replaces the usual spacelike singularity inside the Schwarzschild black hole. The metric ceases to exist at this surface, but all the dynamical fields of the theory are finite. Thus, this surface is only a metric singularity, but not a singularity of the theory. Across this surface, the coordinates t, r change their spacetime roles once more, and one typically finds another Schwarzschild-like horizon behind this "non-metricity" surface. The theory does not admit arbitrarily small black holes: for small objects, the curvature on the would-be horizon is so strong that non-metric modifications prevent the horizon from being formed. Instead of horizon, one has "naked" hypersurface of non-metricity in this case. The details of the conformal diagram depend on the specific shape of the function $\phi(\beta)$. For a simple choice $\phi(\beta) = \ell^2\beta^2$, it is shown in Fig. 1.

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APPENDIX A: PROOF OF THE STATIC PROPERTY OF THE METRIC

In this appendix, we give the details of the proof of static property of the solution to the vacuum spherically symmetric problem.

The most general spherically symmetric expression for the two-form B is given by (14):

$$B = \tilde{X}_- c\, m \wedge l + \tilde{X}_+ c\, n \wedge \bar{m} + \tilde{X}\,(l \wedge n - m \wedge \bar{m}) \tag{A1}$$

with the one-forms l, n, m, and $\bar{m}$ given by (16):

$$l = \frac{1}{\sqrt{2}}\,(f dt - g dr)\ , \quad n = \frac{1}{\sqrt{2}}\,(f dt + g dr)\ , \quad m,\, \bar{m} = \frac{r}{\sqrt{2}}\,(h d\theta \pm i \sin\theta\, d\phi)\ . \tag{A2}$$

Now the functions f, g, h, and c are not assumed to be time-independent, but are functions of both r and t. We are going to show that h is a constant (coordinate-independent), and f, g, and c are time-independent due to the field equations.

The expressions for B are now modified because of the appearance of time derivatives, so that instead of (20), we have

$$\begin{aligned}
dB_- = {} & \frac{1}{\sqrt{2}r}\left[\frac{(rcf)'}{gf}\, l \wedge n \wedge m - c \cot\theta\, l \wedge m \wedge \bar{m}\right] \\
& + \frac{(cg)^{\cdot}}{\sqrt{2}\, fg}\, l \wedge n \wedge m + \frac{nm\dot{h}}{2\sqrt{2}\, fh} l \wedge n \wedge (m + \bar{m})\,, \\
dB_+ = {} & -\frac{1}{\sqrt{2}r}\left[\frac{(rcf)'}{gf}\, l \wedge n \wedge \bar{m} + c \cot\theta\, n \wedge m \wedge \bar{m}\right] \\
& + \frac{(cg)^{\cdot}}{\sqrt{2}\, fg}\, l \wedge n \wedge \bar{m} + \frac{c\dot{h}}{2\sqrt{2}\, fh} l \wedge n \wedge (m + \bar{m})\,, \\
dB_0 = {} & \frac{\sqrt{2}}{rg}\,(l - n) \wedge m \wedge \bar{m} - \frac{\dot{h}}{\sqrt{2}\, fh}\,(l + n) \wedge m \wedge \bar{m}\,,
\end{aligned} \tag{A3}$$

where overdot denotes the time derivative. Formulas (22) are then modified as follows:

$$\begin{aligned}
& (dB_- \wedge m)_v = -\frac{c\dot{h}}{2\sqrt{2}\, fh}\,, \quad (dB_- \wedge \bar{m})_v = \frac{(rcf)'}{\sqrt{2}\, rgf} + \frac{(cg)^{\cdot}}{\sqrt{2}\, fg} + \frac{c\dot{h}}{2\sqrt{2}\, fh}\,, \\
& (dB_+ \wedge m)_v = \frac{(rcf)'}{\sqrt{2}\, rgf} - \frac{(cg)^{\cdot}}{\sqrt{2}\, fg} - \frac{c\dot{h}}{2\sqrt{2}\, fh}\,, \quad (dB_+ \wedge \bar{m})_v = \frac{c\dot{h}}{2\sqrt{2}\, fh}\,, \\
& -(dB_- \wedge n)_v = (dB_+ \wedge l)_v = \frac{c}{\sqrt{2}\, r} \cot\theta\,, \\
& (dB_0 \wedge n)_v = \frac{\sqrt{2}}{rg} - \frac{\dot{h}}{\sqrt{2}\, fh}\,, \quad (dB_0 \wedge l)_v = \frac{\sqrt{2}}{rg} + \frac{\dot{h}}{\sqrt{2}\, fh}\,,
\end{aligned} \tag{A4}$$

the rest of the projections being zero. Then, using formulas (25) and (26), which remain unchanged, we obtain

$$A_- = c^{-1}Pm + cQ\bar{m}\,, \qquad A_+ = cQm + c^{-1}P\bar{m}\,,$$
$$A_0 = \left(c^{-2}P + R\right)(l+n) - \frac{\cot\theta}{\sqrt{2}\,rh}\,(m-\bar{m}) + \frac{\dot{h}}{\sqrt{2}\,c^2 fh}\,l - \frac{\left(cg\sqrt{h}\right)^{\cdot}}{\sqrt{2}\,cfg\sqrt{h}}\,(l-n)\,, \tag{A5}$$

where we made the notation

$$\begin{aligned} P &= -\frac{1}{2\sqrt{2}\,g}\frac{\left(r^2h\right)'}{r^2h} - \frac{\dot{h}}{2\sqrt{2}\,fh}\,, \\ Q &= -\frac{h'}{2\sqrt{2}\,gh} - \frac{\dot{h}}{2\sqrt{2}\,fh}\,, \\ R &= \frac{\left(rcf\sqrt{h}\right)'}{\sqrt{2}\,rcfg\sqrt{h}}\,. \end{aligned} \tag{A6}$$

Now we have to compute the left-hand sides of equations (31) and equate them to zero. Computing the $m \wedge \bar{m}$ component of any of the first two equations in (31), we immediately obtain that $Q = 0$, which, in turn, simplifies the expressions (A5). Calculating then the $(l \wedge \bar{m})$ and $n \wedge \bar{m}$ components of the first equation in (31), we get

$$P\left(\frac{h'}{gh} - \frac{\dot{h}}{fh}\right) = 0\,, \qquad P\left(\frac{h'}{gh} + \frac{\dot{h}}{fh}\right) = 0\,, \tag{A7}$$

respectively. The condition $P = 0$ is excluded because it contradicts the $l \wedge m$ component of the same equation, which contains a generically nonzero expression $\beta + \phi$. Therefore, one must have $h \equiv \mathrm{const}$.

The remaining four equations stemming from the first two equations in (31) read

$$\begin{aligned} &\frac{\left(rc^{-1}P\right)'}{\sqrt{2}\,rg} - \frac{P\dot{g}_*}{\sqrt{2}\,cfg_*} + c^{-1}P\left(c^{-2}P + R\right) - \frac{\left(c^{-1}P\right)^{\cdot}}{\sqrt{2}\,f} = -c(\beta+\phi)\,, \\ &\frac{\left(rc^{-1}P\right)'}{\sqrt{2}\,rg} - \frac{P\dot{g}_*}{\sqrt{2}\,cfg_*} - c^{-1}P\left(c^{-2}P + R\right) + \frac{\left(c^{-1}P\right)^{\cdot}}{\sqrt{2}\,f} = 0\,, \\ &\frac{\left(rc^{-1}P\right)'}{\sqrt{2}\,rg} + \frac{P\dot{g}_*}{\sqrt{2}\,cfg_*} + c^{-1}P\left(c^{-2}P + R\right) + \frac{\left(c^{-1}P\right)^{\cdot}}{\sqrt{2}\,f} = -c(\beta+\phi)\,, \\ &\frac{\left(rc^{-1}P\right)'}{\sqrt{2}\,rg} + \frac{P\dot{g}_*}{\sqrt{2}\,cfg_*} - c^{-1}P\left(c^{-2}P + R\right) - \frac{\left(c^{-1}P\right)^{\cdot}}{\sqrt{2}\,f} = 0\,, \end{aligned} \tag{A8}$$

where $g_* = cg$, as usual. Now, subtracting the first equation from the third one, and the second from the fourth one, we obtain, respectively,

$$\frac{P\dot{g}_*}{cg_*} + \left(c^{-1}P\right)^{\cdot} = 0\,, \qquad \frac{P\dot{g}_*}{cg_*} - \left(c^{-1}P\right)^{\cdot} = 0\,, \tag{A9}$$

which implies

$$\dot{g}_* = 0\,, \qquad \left(c^{-1}P\right)^{\cdot} = 0\,. \tag{A10}$$

These two equations are equivalent in view of the condition $h = \text{const}$ and definition (A6).

Under condition (A10), the remaining two equations stemming from (A8) are precisely equations (32), while the last equation in (31) leads to the two equations (33). Thus, we have the system of differential equations (35). The second equation in (35) then implies that β does not depend on time, hence, by virtue of (15), c is also time-independent. Finally, the third equation in (35) implies that f can only have a time-dependent overall factor, which can always be rescaled to a constant by changing the time variable. This completes the proof of the static property of the spherically symmetric vacuum solution in the theory under investigation.

APPENDIX B: MASS RENORMALIZATION AND REDSHIFT: GENERAL ANALYSIS

In this appendix, we generalize the examples considered in Sec. V. Thus, assume that two regions are characterized by the conditions $\ell^2\beta \ll 1$ and $\ell^2\beta \gg 1$, respectively, where ℓ is some length scale. We will find the approximate solutions in these regions assuming the behavior

$$\phi_\beta = \begin{cases} 2\alpha_1 x^{p_1} + o\left(x^{p_1}\right), & x \ll 1\,, \\ 2\alpha_2 x^{-p_2} + o\left(x^{-p_2}\right), & x \gg 1\,, \end{cases} \tag{B1}$$

where the variable $x = \ell^2\beta$, and α_1, α_2, $p_1 > 0$ and $p_2 > 0$ are different constants.

At small radial distances, where $x \gg 1$, we integrate equations (42) to obtain

$$\frac{\ell^2 r_s}{2r^3} = x\left[1 + \frac{\alpha_2}{p_2} x^{-p_2} + o\left(x^{-p_2}\right)\right], \quad f_* g_* = 1 + \frac{\alpha_2}{p_2} x^{-p_2} + o\left(x^{-p_2}\right), \quad x \gg 1\,. \tag{B2}$$

We also have

$$c^2 = 1 - 3\alpha_2 x^{-p_2} + o\left(x^{-p_2}\right), \quad x \gg 1\,. \tag{B3}$$

Collecting all the terms, we obtain the leading contribution to the g_{00} coefficient in metric (18):

$$g_{00} = f^2 = 1 - \frac{r_s}{r} + \phi_\infty r^2 + 3\alpha_2\left(1 + \frac{2}{3p_2}\right)\left(\frac{2r^3}{\ell^2 r_s}\right)^{p_2}, \qquad r^3 \ll \ell^2 r_s\,, \tag{B4}$$

where $r_s = \text{const}$ is the value of the effective Schwarzschild radius at small distances, ϕ_∞ is the value of the function $\phi(\beta)$ at infinity. We have also assumed the condition $r_s \ll r$, which guarantees that some terms that are of lower power in r are actually subleading. This expression is valid if $p_2 \neq 1$. In the interesting case $p_2 = 1$, the correction to the Schwarzschild–de Sitter metric will also have logarithmic terms; see our concrete example (64) in Sec. V.

As we discussed above, the physical metric can be specified by an additional conformal factor, in which case, we obtain somewhat different corrections. Thus, for the Urbantke metric defined in (52) and for the "volume" metric defined in (54), we will have the coinciding approximate relations for $r^3 \ll \ell^2 r_s$:

$$\tilde{g}_{00} = 1 - \frac{r_s}{r} + \phi r^2 + 2\alpha_2 \left(1 + \frac{1}{p_2}\right)\left(\frac{2r^3}{\ell^2 r_s}\right)^{p_2} . \tag{B5}$$

At large radial distances, where $x \ll 1$, we have

$$\begin{aligned} \frac{\ell^2 r_s}{2r^3} &= \exp\left(\int_0^\infty \frac{\phi_\beta}{2\beta} d\beta\right) x \left[1 - \frac{\alpha_1}{p_1} x^{p_1} + o\left(x^{p_1}\right)\right] , \\ f_* g_* &= \exp\left(\int_0^\infty \frac{\phi_\beta}{2\beta} d\beta\right) \left[1 - \frac{\alpha_1}{p_1} x^{p_1} + o\left(x^{p_1}\right)\right] , \end{aligned} \tag{B6}$$

$$c^2 = 1 - 3\alpha_1 x^{p_1} + o\left(x^{p_1}\right) , \quad x \ll 1 . \tag{B7}$$

In this case, we obtain the following leading contribution to the metric:

$$g_{00} = f^2 = Z^{-2}\left[1 - \frac{Z r_s}{r} + \phi_0 r^2 + 3\alpha_1\left(1 - \frac{2}{3p_1}\right)\left(\frac{\ell^2 Z r_s}{2r^3}\right)^{p_1}\right] , \qquad r^3 \gg \ell^2 Z r_s , \tag{B8}$$

where $\phi_0 = \phi(0)$, and

$$Z = Z(\infty, 0) = \exp\left(-\int_0^\infty \frac{\phi_\beta}{2\beta} d\beta\right) . \tag{B9}$$

In the case of extra conformal factors, again, the corrections to the Schwarzschild-de Sitter metric will be somewhat different. For the Urbantke metric defined in (52) and for the "volume" metric defined in (54), we have the same approximate relations for $r^3 \gg \ell^2 r_s$:

$$\tilde{g}_{00} = Z^{-2}\left[1 - \frac{Z r_s}{r} + \phi_0 r^2 + 2\alpha_1\left(1 - \frac{1}{p_1}\right)\left(\frac{\ell^2 Z r_s}{2r^3}\right)^{p_1}\right] . \tag{B10}$$

We note that $c^2 \approx 1$ in our approximation in these regions, so the physical metric is well defined. The contributions proportional to α_2 and α_1 in (B4), (B5) and (B8), (B10), respectively, are small and can be dropped if the constants α_1 and α_2 are not very large. However, these contributions themselves are of the same order as the "nonmetricity." Their

physical interpretation, therefore, requires the knowledge of the matter couplings in our theory, which is an issue still to be resolved.

For $Z > 1$, the observed gravitational mass of the central object at large distances is Z times larger than it is at small distances. In the intermediate region of radial distances

$$1 \lesssim \frac{r^3}{\ell^2 r_s} \lesssim Z\,, \tag{B11}$$

the metric coefficient behaves in a complicated way; moreover, the physical metric may not be well defined in this region at all. If several regions exist in the space of β in which $\phi_\beta \ll 1$, then obvious renormalizations of the observed gravitational masses and redshift factors exist between these regions.

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